The splitting number and sequential compactness

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Abstract

Topology contains the notions of "compactness" and "sequential compactness". Sequential compactness is more familiar to those who come at the problem via analysis, whileas compactness is a more generally applicable notion. For metric spaces the two are equivalent, but in general neither implies the other.

It turns out that the emergence of compact spaces which are not sequentially compact happens at a precisely defined point of the spaces being "too large". This article introduces some concepts that let us formalize and prove this.

Definition 1. *Let $F \subseteq P(\mathbb{N})$. We say F is splitting if for every infinite $A \subseteq \mathbb{N}$ there is some $B \in F$ such that $A \cap B$ and $A \cap B^c$ are both infinite.*

Clearly there exist splitting sets. e.g. $P(\mathbb{N})$ itself is one. However they can't be too small:

Theorem 2. *Let $F = \{F_1, ...F_n, ...\}$ be a countable collection of infinite sets. F is not splitting.*

Proof. Construct a sequence $A_1 \supseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$ of infinite sets as follows:

Let $A_1 = F_1$.

Having chosen A_n, let B_{n+1} be a choice of one of $A_n \cap F_{n+1}$ and $A_n^c \cap F_{n+1}$ such that B_{n+1} is infinite.

Now, let choose $a_n \in A_n$ and let $A = \{a_n : n \in \mathbb{N}\}$. Then by construction we have for each n, $a_k \in A_n$ or $a_k \in A_n^c$ for all $k > n$, so F does not split A. □

Definition 3. *The splitting number, s, is the smallest cardinality of a splitting family.*

Proposition 4. $\aleph_0 < s \leq 2^{\aleph_0}$

Proof. This is just a restatement of previous results. □

Theorem 5. *The topological space $\{0,1\}^s$ is not sequentially compact.*

Proof. Let F be some splitting family of size s. We will show $\{0,1\}^F$ is not compact.

Let $x_n(f)$ be 1 if $n \in f$ else 0. x_n has no convergent subsequence. Suppose it did. Let $A = \{a_1, ..., a_n, ...\}$ be the indices of that sequence. Then we can find f such that infinitely many members of A are contained in each of f and f^c. For elements $a_k \in f$ we have $x_{a_k}(f) = 1$ and for elements $a_k \in f^c$ we have $x_{a_k} = 0$. But $x \to x(f)$ is a continuous function, so it must converge to some value if x_n were to converge, which is impossible as the sequence contains two subsequences, one where this function converges to 0 and the other where it converges to 1. $\square$

Definition 6. *The weight of a topological space X, written $w(X)$, is the smallest cardinality of a basis for the topology. Equivalently it is the smallest cardinality of a sub-basis for X.*

Lemma 7. *Let X be a T_1 space. $|X| <= 2^{w(X)}$*

Proof. Because X is T_1, every point is an intersection of the open sets containing it, and thus of the basis elements containing it. So if B is a basis, the mapping $x \to \{b : x \in b\}$ is an injection into $P(B)$. $\square$

Theorem 8. *Let κ be a cardinal. $w(\{0,1\}^\kappa) = \kappa$.*

Proof. By the lemma, we have that $w(\{0,1\}^\kappa) \geq \kappa$. It thus suffices to construct a basis of cardinality κ, but the standard basis used in the definition of the product topology has this cardinality. $\square$

And now we connect the two together:

Theorem 9. *Let X be a compact Hausdorff space with $w(X) < s$. X is sequentially compact.*

Proof. Let (x_n) be a sequence, and B some basis with $|B| < s$. For U an open set, let $U' = \{n \in \mathbb{N} : x_n \in U\}$.

Then $\{U' : U \in B\}$ is not a splitting subset of $\mathbb{N}$, so there is some infinite $A = \{a_n\} \subseteq \mathbb{N}$ which has the property that for every $U \in B$, A has finite intersection with either U' or U'^c. i.e. x_{a_n} has finite intersection with U or U^c for every $U \in B$.

Now, consider $Y = \bigcap_{U \in B, x_{a_n} \text{ meets U finitely often}} U^c$.

Y is an intersection of closed non-empty finitely intersecting sets in a compact space, so is non-empty.

Additionally it consists of only one point: Suppose not. Then let U, V be two disjoint basis elements which have non-zero intersection with it. Each of these must contain infinitely many x_{a_n}, but they are disjoint and thus their complements must also contain infinitely many x_{a_n}, contradicting the choice of A.

So, Y consists of exactly one point, say x, and each neighbourhood of that point contains infinitely many elements of x_{a_n}. Thus $x_{a_n} \to x$. $\square$

And thus we have proven the following precise statement about when compact spaces which are not sequentially compact occur:

Theorem 10. *The splitting number* $\mathfrak{s}$ *is the smallest cardinal* κ *such that there is a compact Hausdorff space which is not sequentially compact and has weight* κ.