Counting to Infinity: Ordinals and Transfinite Processes

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Abstract

These notes are based on an online talk that I gave on the same topic. The content is mostly the same, but it's been revised and tidied up to make it more coherent. I've also included some things which I wanted to include but didn't due to incompetence constraints on my part.

1 Introductory Ideas

1.1 A recap on the natural numbers

Before I start on ordinals and transfinite recursion, I will include a brief reminder of the natural numbers and finite recursion.

Recall the peano axioms for $\mathbb{N}$. There is a number $0 \in \mathbb{N}$ and a function $s : \mathbb{N} \to \mathbb{N}$ with $n \to n'$, satisfying the following axioms:

1. $\forall n \;\; n' \neq 0$
2. $\forall m, n \; m' = n' \implies m = n$
3. Suppose $A \subseteq \mathbb{N}$ with $0 \in A$ and $\forall n \in A \;\; n' \in A$. Then $A = \mathbb{N}$.

The last axiom is called the induction axiom.

We have the principle of recursive definition. Suppose X is a set with $x \in X$ and $h : X \to X$. There is a unique function $f : \mathbb{N} \to X$ with $f(0) = x$ and $f(n') = h(f(n))$. This allows us to define the notion of adding two natural numbers, which lets us define the relation $\leq$ on $\mathbb{N}$ and show that this is a total order. In this order n' is the least element greater than n.

It also implies another induction statement, sometimes called 'strong' induction. Suppose $A \subseteq \mathbb{N}$ such that $\forall x \in \mathbb{N} \;\; (\forall y < x \; y \in A \implies x \in A)$.

Proof. Let A be such a set. Let $B = \{x \;\; : \;\; \forall y < x \; y \in A\}$. By hypothesis $B \subseteq A$.

Then $0 \in B$ because the condition is satisfied vacuously. Suppose $n \in B$. Then $y < n' \implies y = n$ or $y < n$. If $y < n$ then $y \in A$ because $n \in B$. Further $n \in B$, so $n \in A$. Thus $\forall y < n'\ y \in A$.

Hence $B = \mathbb{N}$. Thus $\forall n \in \mathbb{N},\ n' \in B$. Hence $n \in A$ as $n < n'$.

$\square$

This then proves that the relation $\leq$ on $\mathbb{N}$ is a well order. Suppose $A \subseteq \mathbb{N}$ has no least element. Then let $n \in \mathbb{N}$. If $\forall x < n\ \ x \in A^c$ then also $n \in A^c$, as else it would be the least element of A. Hence $A^c = \mathbb{N}$, so $A = \emptyset$. Thus every non-empty subset of $\mathbb{N}$ has a least element.

1.2 Some examples of finite recursion

The notion of a recursive definition leads usefully to the notion of an ‘algorithm’, which helps us construct various objects in mathematics. It often isn’t strictly speaking in the form we established, but it’s usually obvious now to convert it into such. e.g. we sometimes say do this until some conditions happens, and then stop. We could make this into the form we introduced in the previous section by adding a halt state to X and letting h send everything to this halt state when the condition is satisfied. We will rarely bother doing this.

Euclid’s algorithm is one example of a useful recursive process on n, for calculating the highest common factor of two natural numbers. You get a decreasing sequence of natural numbers which must be eventually constant because the natural numbers are well ordered.

Here are some other useful examples from algebra:

Let V be a finite dimensional vector space. We want to show that V has a basis.

We will construct an increasing sequence of linearly independent sets B_n. Let $B_0 = \emptyset$. Having picked B_n, if it spans then stop. Else pick some $x \notin \mathrm{span}(B_n)$ and let $B_{n+1} = B_n \cup \{x\}$.

This process must eventually stop because V is finite dimensional. When it stops, we will have produced a basis.

Note that we could have started with B_0 being any linearly independent set, so this actually shows that any linearly independent set is contained in a basis.

Now let R be a noetherian ring with a 1[1]

We will construct a strictly increasing sequence of ideals. Let I_0 be any proper ideal. Having picked I_n, it is not maximal as R has no maximal ideal. So we can pick a proper ideal strictly containing I_n. Thus we have an increasing sequence of ideals, I_n, contradicting the hypothesis that R was noetherian.

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[1] In the seminar I didn’t mention that R needed a 1. As zeno pointed out, this is a mistake and I do actually need it for the result.

Now, suppose we have some infinite group G, and a countable set $X \subseteq G$. We want to show that the group generated by X is countable (e.g. we might want to do this if G were uncountable and we wanted to know that X didn't generate it).

Proof. Define $X_0 = X \cup \{e\}$.

Let $X_{n+1} = X_n \cup \{g^{-1} \;:\; g \in X_n\} \cup \{gh \;:\; g, h \in X_n\}$. Note that if $m \leq n$ then $X_m \subseteq X_n$.

We can easily show by induction that X_n is countable. Thus $H = \bigcup X_n$ is a countable union of countable sets, and so countable.

I claim that H is the group generated by X. Firstly, it is clear that H is contained in the group generated by X (it is a trivial proof by induction to show that for each n, X_n is contained in this group). Thus all we need to do is show that H is a group. Suppose $g, h \in H$. Then $g \in X_n, h \in X_m$ for some m, n. Let $k = \max\{m, n\}$. Then $g, h \in X_k$, so $g^{-1}, gh \in X_{k+1} \subseteq H$. Hence H is a group.

□

1.3 Onto infinite recursion

All these proofs so far have made use of some sort of finiteness condition on the things we were looking at. Can we make a proof like this work in general?

Let's look at the vector space example first. Our algorithm fails to halt, so we get an increasing sequence $B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$. We can take the union of this, $B_\omega = \bigcup B_n$ (the meaning of the ω will become clear later) and get a countably infinite linearly independent set. Is this set a basis? It is fairly clear that it doesn't have to be, even in the simplest cases.

For example suppose we know we have a countably infinite basis for the vector space V, say $e_1, \ldots, e_n, \ldots$. If we happen to be lucky and pick $B_n = \{e_1, \ldots, e_n\}$ then B_ω will indeed be a basis. On the other hand, if we pick $B_n = \{e_2, \ldots, e_{n+1}\}$ then what we get won't be a basis. However we can keep iterating the process and then pick a point not in the span of B_ω and add that in.

If we'd initially picked $B_n = \{e_2, e_4, \ldots, e_2 n\}$ then we'd have to iterate this infinite many times after getting B_ω in order to take a basis, and thus would have ended up with having to take unions twice. A little bit of ingenuity will quickly show that you can make it so you need to take unions infinitely many times. (This is left as an exercise for the interested reader).

So even in the case where we know there is a countable basis this can get extremely complicated, and we're going to have to be clever to make this proof work, let alone what happens when there isn't a countably infinite basis! We're going to need an extra tool to make this work.

A similar problem happens with the maximal ideal example. Suppose our ring is not noetherian, we can once again create that sequence. We can then take the union of it to again get an ideal which is proper (because it does not contain 1) and thus not maximal, so we can carry on. But there no longer seems to be any guarantee that this process will terminate. What can we do to ensure that it does?

Finally, let's look at the group example. Suppose we were wanting to look at something generated by X with a more complicated operation, perhaps one which takes countably many arguments. e.g. a σ-algebra. If the process does not halt then we can pick $x_n \in X_{n+1} \setminus X_n$ and then consider $\bigcup x_n$ and we have no guarantee that this will be in the union.

All these sorts of things are the type of example that will motivate our initial construction of the ordinals.

2 An introduction to the Ordinals

2.1 The Axioms

We shall define the ordinals by an axiomatisation similar to the peano axioms, but with an extra operation that corresponded to what we were doing when we 'took the limit' when the process didn't halt as we iterated it.

For the natural numbers we defined the order in terms of the succesor operation. For the ordinals we will do it the other way around, because it will turn out that not every ordinal is the succesor of some other ordinal.

We will adopt the standard convention of using greek letters for ordinals. I didn't do this in the seminar, but in writing these notes it got incredibly irritating to not do it.

Denote the collection of all ordinals by Ord. We have an ordinal $0 \in Ord$, a relation $\leq$ on the ordinals and a function $\alpha \to \alpha'$ on the ordinals.

The following are our axioms for the ordinals:

1. $\leq$ is a total order.
2. 0 is the least ordinal.
3. For any ordinal α, α' is the least ordinal greater than α.
4. For any *set* of ordinals A, there is an ordinal $\bigvee A$ which is the least upper bound of A.
5. Let K be a collection of ordinals which contains 0 and satisfies $\forall \alpha \in K \;\; \alpha' \in K$ and for every subset A of K, $\bigvee A \in K$. Then $K = \text{Ord}$.

The last is our analogue of the induction axiom. $\bigvee$ is the operation that we need to capture our notion of taking limits.

2.2 A digression on classes and sets

We now make an important observation: There cannot be a set of all ordinals. If there were then there would be an ordinal $\alpha = \bigvee \text{Ord}$. This would neccesarily be the largest ordinal. But $\alpha' > \alpha$, so α was not the largest ordinal after all. Contradiction. This feature of the ordinals is called the Burali-Forti paradox.

This should put you in mind of Russel's paradox, that there is no set of all sets. Essentially the idea is that certain collections of objects are 'too big' to actually be sets. A collection which is not a set is called a proper class. I will not be terribly formal about the distinction between sets and proper classes except where I have to be. In particular I shall not worry about the set theoretic details of defining functions on proper classes, etc. and will assume it works without worrying about it unduly.

I will need to prove that various things are sets, but because I am doing this informally all I'm really doing is showing you that under a reasonable axiomatisation of set theory we would expect them to be sets. One thing I will note (this is actually an axiom if we do it formally) is that if a collection bijects with a set, then it must be a set. This is because of our intuitive notion that proper classes are collections which are too big to be sets.

This gives us an important result:

Lemma 2.1. *There is no injective function from the ordinals into a set.*

Proof. Suppose we had an injection $f : \text{Ord} \to X$. If there were then the ordinals would biject with their image $f(X)$, which is a subset of X. This is a contradiction. ☐

This is what we will use to conclude that a lot of processes we define on the ordinals will terminate.

2.3 Basic properties of Ord

We begin with some basic definitions about ordinals:

Lemma 2.2. *Let α be an ordinal. The collection of ordinals β with $\beta < \alpha$ is a set.*

Proof. Let A be the collection of ordinals α such that the ordinals $< \alpha$ form a set.

Then $0 \in A$, as the collection of ordinals less than 0 is empty.

Suppose $\alpha \in A$ and $\beta < \alpha'$. Then either $\beta < \alpha$ or $\beta = \alpha$. Thus the collection of ordinals less than α' is $\{\beta \ : \ \beta < \alpha\} \cup \{\alpha\}$.

Suppose S is a set of ordinals in A. Let $\alpha = \bigvee S$. Then if $\beta < \alpha$ we have $\beta \leq \tau$ for some $\tau \in S$, as α was the least upper bound of S. So the collection of ordinals $< \alpha$ is contained in $\bigcup_{\tau \in S} \{\beta \ : \ \beta \leq \tau\}$, which is a union of a set of sets and thus a set. Hence the collection of ordinals less than α is contained in some set and so itself a set.

□

Definition 2.1. 1. *Let α be an ordinal. If there exists β with $\alpha = \beta'$ then α is said to be a succesor ordinal.*

2. *If α is a non-zero ordinal which is not a succesor it is said to be a limit ordinal.*

3. *A set of the form $\{\beta \in Ord \ : \ \beta < \alpha\}$ is called an initial segment.*

Now, define a function $f : \mathbb{N} \to \text{Ord}$ by $f(0) = 0$ and $f(n') = f(n)'$. This is trivially an isomorphism of $\mathbb{N}$ onto its image. We can thus identify $\mathbb{N}$ with a subset of the ordinals. Define $\omega = \bigvee \mathbb{N}$.

Lemma 2.3. *ω is a limit ordinal and $\mathbb{N}$ is the initial segment $\{\alpha \ : \ \alpha < \omega\}$*

Proof. Suppose $\beta \notin \mathbb{N}$. Then $\forall n \in \mathbb{N} \ \ n < \beta$. The proof of this is by induction on n. $0 < \beta$. If $n < \beta$ then so is n', as $n' \neq \beta$ (else we would have $\beta \in \mathbb{N}$. Hence $\omega \leq \beta$. Thus if $\beta < \omega$ we must have $\beta \in \mathbb{N}$. Further, $\mathbb{N} \subseteq \{\beta \ : \ \beta \leq \omega\}$ because ω is an upper bound for $\mathbb{N}$, but clearly $\omega \notin \mathbb{N}$ as $\omega' \notin \mathbb{N}$ and $\mathbb{N}$ is closed under $'$. Hence $\mathbb{N} = \{\beta \ : \ \beta < \omega\}$.

This also shows that ω is a limit ordinal. If $\alpha < \omega$ then $\alpha \in \mathbb{N}$, so $\alpha' \in \mathbb{N}$ and thus $\alpha' \neq \omega$.

□

We also prove the analogous form of strong induction:

Lemma 2.4. *Let A be a collection of ordinals such that $\forall \alpha \in Ord \quad (\forall \beta < \alpha \ \ \beta \in A \implies \alpha \in A)$. Then $A = Ord$.*

Proof. The first part of this proof is identical to that for $\mathbb{N}$. We merely need to show that the set B is closed under taking $\bigvee$.

Suppose we have a set S of ordinals in B. Let $\alpha = \bigvee S$. If $\beta < \alpha$ then $\beta \leq s$ for some $s \in S$. So either $\beta < s$ and thus $\beta \in A$, or $\beta = s$, and thus once again $\beta \in A$. Hence $\bigvee S \in B$. □

Lemma 2.5. *Ord is well ordered by $\leq$.*

Proof. The proof is identical for that of $\mathbb{N}$. □

We will now make a few more definitions and then give a form of the recursion theorem for the ordinals.

Definition 2.2. *Let X be a set and a an ordinal. A sequence of length a is a function $f : \{t : t < a\} \to X$. We denote the set of sequences of length a by $X^{(a)}$. We denote the class of sequences of any ordinal length by $X^{(Ord)}$. (Obviously this is not a set).*

Theorem 2.1. *Suppose we have a non-empty set X and a function $h : X^{(Ord)} \to X$.*

There is a unique function $f : Ord \to X$ with $f(a) = h(f|_{\{t\ :\ t<a\}})$.

(The function $f|_{\{t\ :\ t<a\}}$ is exactly what we mean by a sequence of length a).

Proof. I'm only going to sketch a proof of this. Let B be the class of ordinals for which there exists a unique function on $\{t : t \leq \alpha\}$ satisfying this condition. Show by strong induction that $B = Ord$. Let f_α be the unique function on $\{t : t \leq \alpha\}$ satisfying the condition. Define $f(\alpha) = f_\alpha(\alpha)$. This satisfies the condition, and then induction shows that it's unique. ☐

Note that this actually works on any well ordered set, not just the ordinals. In particular we will want to use it later on initial segments of the ordinals.

2.4 Some Basic Applications

We will now use ordinals to prove the first two of our algebraic results in generality.

Theorem 2.2. *Every vector space has a basis.*

Proof. Let V be a vector space. We define a function B from the ordinals into the linearly independent subsets of V.

Let $B_0 = \emptyset$. Given a sequence a sequence of length a define B_a as follows:

If $a = b'$ for some b, i.e. a is a succesor ordinal, consider B_b. If B_b spans then it is a basis, so we're done and stop the process.

If B_b does not span then we can pick some α not in span(B_b). Let $B_a = B_b \cup \{\alpha\}$.

If a is a limit ordinal define $B_a = \bigcup_{b<a} B_b$.

Note that by construction for any two ordinals if $b < a$ then $B_b \subsetneq B_a$. Hence this process must stop, for if it hasn't then we have constructed an injective function from the ordinals into $P(V)$, which is impossible. Thus we produce a basis of V.

☐

This demonstrates that the ordinals do indeed suitably capture our notion of iterating the process for long enough to produce the desired result.

Note what we did in this proof: In defining our recursion we conditioned on whether b was 0, a succesor or a limit. This is very standard and is what will be done in almost all proofs of this sort.

The proof of the result about ideals is identical. Start with $I_0 = 0$. At succesors extend if it's not maximal, else stop. At unions take limits. This process must stop.

Note that both of these proofs are ones we would normally do with zorn's lemma. In fact as we shall see shortly, the proofs are really identical to the proof via zorn's lemma.

Lemma 2.6. *Let X be a partially ordered set in which every chain has an upper bound (a chain is a subset totally ordered by the inherited relation). X has a maximal element.*

Proof. Define an ordinal sequence of chains C_t by:

$C_0 = \{\alpha\}$ for some $\alpha \in X$.

If C_t has no greatest element then let $C_{t'} = C_t \cup \{y\}$ where y is some upper bound for C_t. This is still a chain.

If C_t has a greatest element, say y, then if y is a maximal element of X we stop. Else Let $C_{t'} = C_t \cup \{y_2\}$ where $y < y_2$. This is again still a chain.

If t is a limit ordinal let $C_t = \bigcup\limits_{s<t} C_s$.

This process must stop. When it does we will have produced a maximal element.

□

So really all that we're doing is reproducing the proof of zorn's lemma in the above examples. However, given that the proof of zorn's lemma from the ordinals is so trivial it is perhaps fairer to say that the two methods are directly equivalent. Further there are some examples (as we'll see later) where you want more control than zorn's lemma gives, so that transfinite induction does prove genuinely different from using zorn.

2.5 Cardinals

Lemma 2.7. *Let X be any set. There is an ordinal α such that X bijects with $\{\beta \ : \ \beta < \alpha\}$.*

Proof. Same ol' proof by transfinite recursion. We define an injective function on the ordinals into X.

If $f|_{\{s \ : \ s<t\}}$ is surjective, then stop. We have a bijection from an initial segment to X.

Else it is not surjective, so we may pick some x not in the image and declare $f(t) = x$. Then $f|_{\{s \,:\, s<t\}}$ is still injective.

This process must terminate, etc.

$\square$

This tells us two important things: It tells us that every set can be well ordered, which is nice, and it tells us that we can find an initial segment of Ord of every cardinality.

Given any set X we can now use the ordinals to pick a particular representative of the sets which biject with it. We define the cardinality of X to be the least ordinal β such that $\{t : t < \beta\}$ bijects with X. Ordinals of this form are called cardinals.

Two particularly interesting cardinals are c, the cardinality of R, and ω_1, the least ordinal x such that $\{t : t < \alpha\}$ is uncountable.

Note that for two sets X, Y with cardinals α, β we have $|X| < |Y|$ iff $\alpha < \beta$. Hence we now know that $<$ is a total order on cardinalities.

Note that any non-zero cardinal is a limit.

Definition 2.3. *A countable ordinal is an ordinal* $< \omega_1$.

Lemma 2.8. *Let* A *be a set of countable ordinals. Then* $\bigvee A < \omega_1$.

Proof. Certainly $\bigvee A \leq \omega_1$. Suppose $\bigvee A = \omega_1$. Then if $\alpha < \omega_1$ we have $\alpha < \beta$ for some $\beta \in A$. So $\{\alpha \;:\; \alpha < \omega_1\} = \bigcup\limits_{\beta \in A} \{\alpha \;:\; \alpha < \beta\}$. But each of the sets $\{\alpha \;:\; \alpha < \beta\}$ is countable, so $\{\alpha \;:\; \alpha < \omega_1\}$ is a countable union of countable sets, and thus itself countable. But it is uncountable by hypothesis, so this is a contradiction.

$\square$

2.6 Some more applications of transfinite recursion

Our next application is an application merely of transfinite recursion on the ordinals up to ω_1.

Theorem 2.3. *Let* B *be the borel* $\sigma-$*algebra on* $\mathbb{R}$. *i.e. the sigma-algebra generated by* $\{[a, b] \;:\; a < b, a, b \in \mathbb{R}\}$. *Then* $|B| = |R| (= c)$.

Proof. We will construct an ω_1 sequence of sets of subsets of $\mathbb{R}$ that produce the borel subsets in a similar way that our example of generating a group did.

Define $B_0 = \{[a, b]\} \cup \{\emptyset, \mathbb{R}\}$.

Having defined B_α, define $B_{\alpha'}$ by $B_{\alpha'} = B_\alpha \cup \{A^c \;:\; A \in B_\alpha\} \cup \{\bigcup A_n \;:\; A_1, \ldots, A_n, \ldots \in B_\alpha\}$.

At a limit ordinal $\alpha < \omega_1$ define $B_\alpha = \bigcup\limits_{\beta<\alpha} B_\beta$.

Define $H = \bigcup_{\alpha<\omega_1} B_\alpha$. I claim $B = H$

Again, it is clear that $B \supseteq H$. It only needs to be shown that H is a $\sigma-$algebra.

It is clear that if $A \in H$ then A^c in H, by a similar proof to the group one.

Suppose $A_1, \ldots, A_n, \ldots \in H$. Then we can find α_n with $A_n \in B_{\alpha_n}$. Then by lemma 2.8 $\alpha = \bigvee\{\alpha_n\} < \omega_1$. So $A_1, \ldots, A_n, \ldots \in B_\alpha$. So $\bigcup A_n \in B_{\alpha'} \subseteq H$. This proves the claim.

Now, it is an easy proof by transfinite induction that $|B_\alpha| = |\mathbb{R}|$, using the fact that $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$. Hence $|H| \leq |R|\omega_1$ (Where we are considering ω_1 as a cardinal and this is cardinal multiplication.). Thus $|H| \leq |\mathbb{R}|\omega_1 \leq |\mathbb{R}||\mathbb{R}| \leq |R^2| = |\mathbb{R}|$. Further it is clear that $|\mathbb{R}| \leq |H|$, e.g. as $\forall x \in \mathbb{R}, [x, x+1] \in H$. Thus $|H| = |\mathbb{R}|$.

□

This for example shows that there are lebesgue measurable sets which are not borel measurable, as every subset of the cantor set is lebesgue measurable so there are $2^{|\mathbb{R}|}$ lebesgue measurable sets.

We prove another theorem of real analysis / topology by recursion on the ordinals. While in this case the function will be defined on all ordinals the theorem will hinge on showing that there is some countable ordinal after which it is constant - i.e. it is important not merely that the process halts, but also *where* it halts.

Definition 2.4. *$X \subseteq \mathbb{R}$ is said to be perfect if X is closed and every $x \in X$ is a limit point of X. e.g. $[0,1]$ is perfect, as is the cantor set, but $\{0,1\}$ or $\{0\} \cup \{\frac{1}{n} \ : \ n \in \mathbb{N} \ \ n > 0\}$.*

Theorem 2.4. *Cantor-Bendixson Theorem*

Let F be a closed subset of $\mathbb{R}$. We can write $F = P \cup C$ with P perfect and C countable.

Proof. We shall construct a decreasing sequence of closed subsets of $\mathbb{R}$.

Let $F_0 = F$.

Having defined F_α, let $F_{\alpha'}$ be the set of limit points of F_α. This is a closed subset of F_α. At limit ordinals take intersections.

Note that for any α we that have $F_\alpha \setminus F_{\alpha'}$ is a discrete subset of $\mathbb{R}$ and so must be countable.

Further note that for the usual reasons (injective function from the ordinals into $P(\mathbb{R})$) this process must eventually produce a set F_α with $F_\alpha = F_{\alpha'}$. Further such a set must be perfect.

Thus if we can prove that there is a countable ordinal α with $F_\alpha = F_{\alpha'}$ then we're done, as we must have that $F_0 \setminus F_\alpha$ is a countable union of countable sets, and so itself countable, proving the theorem. Thus it only remains to show that this must happen before ω_1.

I claim that there is no strictly decreasing ω_1 sequence of closed subsets of $\mathbb{R}$. By taking complements this is equivalent to showing there is no strictly increasing sequence of open subsets of $\mathbb{R}$.

Proof:

Let U_α be a strictly increasing ω_1 sequence of open subsets of $\mathbb{R}$. Because it is strictly increasing we can pick a point $y_\alpha \in U_{\alpha'} \setminus U_\alpha$. Let B_α be an interval of the form (a, b) with a, b rational and $y_\alpha \in B_\alpha \subseteq U_{\alpha'}$. Suppose $\alpha < z$. Then $B_\alpha \neq B_z$, as $B_\alpha \subseteq U_{\alpha'}$ and $y_z \in B_y$ but also $y_z \notin U_{\alpha'}$ by construction.

So we've constructed an injective function from $\{\alpha : \alpha < \omega_1\}$ into the set of intervals with rational end-points. But this is impossible because the set of intervals with rational endpoints is countable.

This proves the claim, and hence the result.

$\square$

One reason why Cantor-Bendixson is a nice result is that it can be shown that any non-empty perfect subset of $\mathbb{R}$ has cardinality c, because you can stick a copy of the cantor set inside it, so this shows that every closed subset of $\mathbb{R}$ is countable or has cardinality c. This limits the types of sets that a counterexample to the continuum hypothesis can be.

Our remaining two results are amusing but pointless examples of the sort of construction we can do with transfinite induction. These aren't intended to be representative of how useful it is, because neither of them has much use at all, but merely of the sort of ideas and techniques involved. They're also kindof fun.

Theorem 2.5. *Let* $n > 1$. *There is a set* $A \subseteq \mathbb{R}^n$ *such that for any straight line* L *in* $\mathbb{R}^n$, $|L \cap A| = 2$.

Proof. Consider the set of all such straight lines in $\mathbb{R}^n$. This clearly has cardinality c, so biject it with the initial segment corresponding to c to get a sequence of length c, L_α. We will recursively construct a set A_α for $\alpha < c$ such that $\forall \beta \leq \alpha \; |A_\alpha \cap L_\beta| = 2$, and no three points of A_α are colinear.

Suppose we have done so for $z < \alpha$. Consider $B = \bigcup_{\beta < \alpha} A_\beta$. No three points of B are colinear, so $|L \cap B| \leq 2$. If $|L \cap B| = 2$ then define $A_\alpha = B$. This clearly satisfies the conditions.

Suppose $|L \cap B| < 2$. Consider $C = \bigcup \{L \; : \; L$ is a line passing through x and $y, \; x, y \in B \;\; x \neq y\}$.

For each L in the union we have $|L \cap L_\alpha| < 2$, as any lines which share two distinct points are equal and L does not contain two points from B. Thus

$|C \cap L_\alpha| \leq |\{(a,b) \ : a, b \in B, a \neq b\}| \leq |B|^2 < |R|$. Thus $C \cap L_\alpha \neq L_\alpha$.

If $L \cap B$ contains 1 point then we can add in any point t in $L_\alpha \cap C$ to B. Then still no 3 points of $\{t\} \cup B$ are colinear, as t is by construction not on any line between two points of B. Further now $L \cap (B \cup \{t\})$ consists of two points. Define $A_\alpha = B \cup \{t\}$.

If $L \cap B = \emptyset$ then add in one point $\{t\}$ as above, and repeat the previous argument replacing B with $B \cup \{t\}$.

Now, having constructed all these A_α, the set $A = \bigcup_{\alpha < c} A_\alpha$ is our desired set. $\square$

The above is a very good illustration of the types of arguments we often want to make when doing transfinite induction. Our initial segments have a strictly smaller cardinality than the whole set, so what we've done so far can't have filled up the entire set. Thus there are still points left to pick to carry on our construction.

The following is a more elaborate version of the same sort of idea:

Theorem 2.6. *$\mathbb{R}^3$ can be written as the union of a disjoint collection of unit circles.*

Proof. We will enumerate $\mathbb{R}^3$ as $\{x_\alpha \ : \ \alpha < c\}$ and construct a sequence C_α of unit circles in $\mathbb{R}^3$ of length c. Obviously we cannot guarantee that $x_\alpha \in C_\alpha$, but we shall choose C_α so that $x_\alpha \in \bigcup_{\beta \leq \alpha} C_\beta$. Further we shall do so such that $C_\alpha \cap \bigcup_{\beta < \alpha} C_\beta = \emptyset$.

Obviously if we can do this then the set $\{C_\alpha \ : \ \alpha < c\}$ will be our desired covering of $\mathbb{R}^3$ by disjoint unit circles.

Suppose for every $\beta < \alpha$ we have constructed the circles C_β satisfying that condition.

Note that $\bigcup_{\beta < \alpha} C_\beta$ is not all of $\mathbb{R}^3$. This is because if we have an infinite straight line L then $L \cap C_\beta$ has at most two points, so $|L \cap \bigcup_{\beta < \alpha} C_\beta| \leq |\{\beta \ : \ \beta < \alpha\} < c$. So $L \cap \bigcup_{\beta < \alpha} C_\beta \neq L$.

We pick $x \notin \bigcup_{\beta < \alpha} C_\beta$. If $x_\alpha \in \bigcup_{\beta < \alpha} C_\beta$ then pick $x = x_\alpha$, else let x be any other point. We will construct a C with $x \in C$ and $C \cap L \cap \bigcup_{\beta < \alpha} C_\beta = \emptyset$.

Pick a plane P which contains x and does not contain any of the circles C_β. We can do this because for any β there is at most one plane containing C_β, so there are only $|\{\beta \ : \ \beta < c\}| < c$ planes containing one of the circles, and there are c many planes containing x. Each C_β now only intersects P in at most two points, so $T = P \cap \bigcup_{\beta < \alpha} C_\beta$ has fewer than c points.

Given a line L passing through x there are exactly two unit circles tangent to L and containing x. Further any one of these circles determines L uniquely.

For each $y \in T$ there are at most two unit circles in P which contain both x and y. Thus for any $y \in T$ there are only a finite number of lines L passing through x for which a unit circle in P tangent to L and passing through x also passed through y. Hence there are fewer than c lines passing through L such that a unit circle in P tangent to L intersects T. But there are c many lines through x in P, and so we may pick a line L through x and a circle C tangent to L and containing x which does not intersect T, and thus does not intersect $\bigcup_{\beta<\alpha} C_\beta$. We now define $C_\alpha = C$, and this satisfies the conditions.

□

One reason this result is of mild interest is that it is known that you can't do any such thing for the plane. Also, while there are explicit examples of being able to write $\mathbb{R}^3$ as a union of disjoint circles if you allow the radius to vary, there seems to be no obvious way to do the unit circle version explicitly.