

Some Lemmas

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The following lemma comes up when defining topological entropy.

Lemma 1.

Let a_n be a sequence of non-negative real numbers such that

$$\forall m, n \quad a_{m+n} \leq a_m + a_n$$

Then

$$\frac{a_n}{n} \rightarrow \inf_{n \geq 1} \frac{a_n}{n}$$

Proof. Let $I = \inf_{n \geq 1} \frac{a_n}{n}$.

Note that $\frac{a_{kn}}{kn} \leq \frac{a_n}{n}$. So, suppose $\frac{a_n}{n} \leq I + \epsilon$. Then whenever $n|m$ we have $\frac{a_m}{m} \leq I + \epsilon$.

So, suppose $\epsilon > 0$. We can find N such that $\frac{a_N}{N} \leq I + \frac{1}{2}\epsilon$, and thus for all n such that $N|n$ we have $\frac{a_n}{n} < I + \frac{1}{2}\epsilon$.

Let $n \geq N$. We may write $n = kN + r$, where $0 \leq r < N - 1$. So

$$\begin{aligned} \frac{a_n}{n} &\leq \frac{a_{kN}}{n} + \frac{a_r}{n} \\ &\leq \frac{a_{kN}}{kN} + \frac{Na_1}{n} \\ &< \frac{1}{2}\epsilon + \frac{Na_1}{n} \end{aligned}$$

Thus picking N' big enough such that when $n \geq N'$ we have $\frac{Na_1}{n} < \frac{1}{2}\epsilon$, whenever $n \geq \max\{N, N'\}$, we have $I \leq \frac{a_n}{n} \leq I + \epsilon$ and so $|I - \frac{a_n}{n}| < \epsilon$.

Hence $\frac{a_n}{n} \rightarrow I$.

□

The following is a slight generalisation of the well known Cesaro summation method. The normal version is simply the case $b_n = n$.

Lemma 2 (Cesaro's Lemma). *Let b_n be a monotonic increasing sequence of positive real numbers such that $b_n \rightarrow \infty$. Let $x_n \rightarrow x$. Then,*

$$\frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \rightarrow x \text{ as } n \rightarrow \infty$$

Proof. Let $\epsilon > 0$. Pick N such that whenever $n \geq N$ we have $x_n \geq x - \epsilon$.

So for $n \geq N$ we have

$$\begin{aligned} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k &= \frac{1}{b_n} \sum_{k=1}^N (b_k - b_{k-1})x_k + \frac{1}{b_n} \sum_{k=N+1}^n (b_k - b_{k-1})x_k \\ &\geq \frac{1}{b_n} \sum_{k=1}^N (b_k - b_{k-1})x_k + \frac{1}{b_n} \sum_{k=N+1}^n (b_k - b_{k-1})(x - \epsilon) \\ &= \frac{1}{b_n} \sum_{k=1}^N (b_k - b_{k-1})x_k + \frac{1}{b_n} (b_n - b_N)(x - \epsilon) \end{aligned}$$

Hence we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k &\geq \frac{1}{b_n} \sum_{k=1}^N (b_k - b_{k-1})x_k + \frac{1}{b_n} (b_n - b_N)(x - \epsilon) \\ &= 0 + x - \epsilon \\ &= x - \epsilon \end{aligned}$$

So

$$\liminf_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \geq x$$

But similarly we have

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \leq x$$

So

$$x \leq \liminf_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \leq \limsup_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \leq x$$

Thus

$$\liminf_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k = \limsup_{n \rightarrow \infty} \frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k = x$$

So

$$\frac{1}{b_n} \sum_{k=1}^n (b_k - b_{k-1})x_k \rightarrow x$$

□

Lemma 3 (The Dini Lemma). *Let X be a compact topological space and $f_n : X \rightarrow \mathbb{R}$ be a monotonic decreasing sequence of non-negative continuous functions such that $f_n \rightarrow f$ pointwise. Then*

$$\sup_{x \in X} f_n(x) \rightarrow \sup_{x \in X} f(x)$$

(Note that continuity of f is not assumed).

Proof.

Note that

$$\sup_{x \in X} f_n(x)$$

is monotone decreasing and bounded below, so it converges. Say

$$\sup_{x \in X} f_n(x) \rightarrow M$$

Suppose $t < M$.

Let $L_n = \{x \in X : f_n(x) \geq t\}$.

Then L_n is closed. It is non-empty because $\sup_{x \in X} f_n(x) \geq M > t$. Because the sequence f_n is monotone decreasing we have L_n is as well (with respect to \subseteq). Thus by compactness $\bigcap L_n \neq \emptyset$. Let $x \in \bigcap L_n$. Then $\forall n$ $f_n(x) \geq t$. Thus $f_n(x) \geq t$.

Hence

$$\forall t < M, \sup_{x \in X} f(x) \geq t$$

Thus

$$\sup_{x \in X} f(x) \geq M$$

But

$$\forall x \ f(x) \leq f_n(x)$$

Hence

$$\sup_{x \in X} f(x) \leq \sup_{x \in X} f_n(x) \rightarrow M$$

Thus

$$\sup_{x \in X} f(x) = M$$

and so

$$\sup_{x \in X} f_n(x) \rightarrow \sup_{x \in X} f(x)$$

□

Corollary 3.1 (Dini's Theorem). *Let X be a compact topological space and $f_n : X \rightarrow \mathbb{R}$ be a monotonic decreasing sequence of non-negative continuous functions such that $f_n \rightarrow f$ pointwise with f continuous. Then $f_n \rightarrow f$ uniformly.*

Proof.

Apply the lemma to $f_n - f$.

□

Lemma 4 (Abel's Theorem). *If $\sum a_n = l$ then $\sum a_n t^n \rightarrow l$ as $t \rightarrow 1^-$.*

Proof. Define $x_n = \sum_{k=0}^n$. Then, taking $x_{-1} = 0$, we have:

$$\begin{aligned}
f(t) &= \sum a_n t^n \\
&= \sum (x_n - x_{n-1}) t^n \\
&= \sum x_n t^n - \sum x_{n-1} t^n \\
&= (1-t) \sum x_n
\end{aligned}$$

Define $\alpha = \sum a_n$ and $r_n = \sum_{k \geq n} a_k$.

Note that $(1-t) \sum t^n = 1$.

We have

$$\begin{aligned}
\alpha - f(t) &= (1-t) \sum (\alpha - x_n) t^n \\
&= (1-t) \sum r_n t^n \\
|\alpha - f(t)| &\leq (1-t) \sum_{n=0}^N |r_n| t^n + (1-t) \sum_{n=N+1}^{\infty} |r_n| t^n
\end{aligned}$$

Fix $\epsilon > 0$. Now choose N such that $r_{N+1} < \frac{1}{2}\epsilon$. We thus have that

$$\begin{aligned}
|\alpha - f(t)| &\leq (1-t) \sum_{n=0}^N |r_n| t^n + \frac{1}{2}\epsilon \\
&\leq (1-t)M + \frac{1}{2}\epsilon
\end{aligned}$$

Thus for $t > \frac{1}{2M}\epsilon$ we have $|\alpha - f(t)| < \epsilon$.

Hence $f(t) \rightarrow \alpha$ as desired. □

Corollary 4.1. *Let f be an analytic function with a Taylor series of about zero of $\sum a_n z^n$, with radius of convergence R . Suppose $|z| = R$ and $\sum a_n z^n$ converges. Then $\sum a_n z^n = f(z)$.*

Proof. Apply Abel's theorem and note that $\sum a_n z^n t^n = f(tz)$. By continuity $f(tz) \rightarrow f(z)$ as $t \rightarrow 1^-$. □