Filters in Analysis and Topology

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Abstract

The study of filters is a very natural way to talk about convergence in an arbitrary topological space, and carries over nicely into metric spaces with subjects like completeness finding an elegant representation in terms of filters.

The purpose of this paper is to provide a brief discussion of this general theory of filters, and attempt to demonstrate why they are an interesting and useful way to talk about convergence. It is not meant to be a complete introduction to the theory, but merely to provide enough background in it to give the reader a good idea of the essentials.

1 Filters and Convergence

Let X be a topological space, and consider a point $x \in X$. Recall that we define a set V to be a neighbourhood of x if there is an open set U such that $x \in U \subseteq V$.

Let N_x be the set of all neighbourhoods of a x. It is trivial to verify the following properties:

1. $X \in N_x$
2. If $V \in N_x$ and $V \subseteq W$ then $W \in N_x$
3. If $U, V \in N_x$ then $U \cap V \in N_x$
4. $\emptyset \notin N_x$

If F is a collection of subsets of X which satisfy the above properties then we call it a *filter*. N_x is called the neighbourhood filter of x. Note that filters are closed under finite intersection as well as pairwise (by induction).

An important example of a filter is the cofinite filter. Let X be an infinite set; then $F = \{U \subseteq X \ : \ X \setminus U \text{ is finite } \}$

Before we start discussing filters and convergence, we will want to prove and define various things about filters.

To start with, lets consider ways of generating filters. It is an easy check that if C is a collection of filters on X then $\bigcap C$ is also a filter on X. So if there is a filter containing a collection of subsets of X, G then there is a least such filter, F. We will then say that F is the filter generated by G.

Clearly if there exist $U_1, \ldots, U_n \in G$ with $\bigcap_{i=1}^n U_i = \emptyset$ then G cannot generate a filter, as the U_i would be in any filter containing G, so the emptyset would be in that filter, which is a contradiction.

Proposition 1 *Let G be a collection of subsets of X such that for all $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in G$ we have $\bigcap_{i=1}^n U_i \neq \emptyset$. Then $F = \{U \subseteq X \ : \ \exists U_1, \ldots, U_n \in G \quad \bigcap_{i=1}^n U_i \subseteq U\}$. F is a filter containing G. Indeed F is the filter generated by G.*

The proof of this is an easy exercise.

Thus a set G generates a filter iff it satisfies this condition. Such a G is called a subbase for F. If G is closed under finite (pairwise) intersection then it is called a base for F and F takes the simpler form $F = \{U \subseteq X \ : \ \exists V \in G \ \ V \subseteq U \ \}$

Definition 1 *Let F be a filter on X. F is said to be an ultrafilter if for all $A \subseteq X$ either $A \in F$ or $A^c \in F$.*

For example, if $x \in X$ and $F = \{U \subseteq X \ : \ x \in U\}$ then F is an ultrafilter. However, we cannot actually explicitly write down any other ultrafilters; this is admittedly a point where the study of filters loses some of it's naturalness, but fortunately much of the theory does not depend on ultrafilters. In light of this, the following result is perhaps extremely surprising:

Theorem 1 *(The Ultrafilter Theorem) Let F be a filter on X. There is an ultrafilter $\mathbb{U}$ such that $F \subseteq \mathbb{U}$*

Proof

The proof of this is actually quite easy. Consider the set of all filters on X which contain F, together with the partial ordering $\subseteq$.

Let C be a chain in this set. $\bigcup C$ is closed under pairwise intersection, so forms a base for a filter. This filter is then an upper bound for C. Hence every chain has an upper bound, so Zorn's Lemma gives us a maximal element of the set, $\mathbb{U}$. It is then easy to check that a maximal filter must be an ultrafilter. So $\mathbb{U}$ is an ultrafilter containing F.

QED

This does not contradict our original assertion that ultrafilters other than the one generated by a single point cannot be described - the use of the axiom of choice means that the ultrafilter theorem is highly non-constructive. However it does prove the existence of such an ultrafilter - let F be an ultrafilter containing the cofinite filter. Then F cannot be generated by $\{a\}$, as $X \setminus \{a\} \in F$. Such an ultrafilter is called *free.* It is worth noting that any ultrafilter which is not free is generated by a singleton, as if there exists an a with $X \setminus \{a\} \notin F$ then $\{a\} \in F$. The complements of singletons generate the cofinite filter, so if F is not generated by a singleton then it contains the cofinite filter and is thus free.

We can now start discussing convergence in terms of filters.

Definition 2 *Let F be a filter and $x \in X$. We say that F converges to x, or that x is a limit of F if $N_x \subseteq F$. We shall write $F \to x$ to mean F converges to x.*

We shall use the following theorem to demonstrate why this is a useful definition to consider:

Theorem 2 *Let X be a topological space. X is hausdorff iff every filter has at most one limit.*

Proof:

Suppose X is hausdorff and let $x \neq y$. Then there are neighbourhoods U and V of x and y respectively with $U \cap V = \emptyset$. Thus no filter contains both U and

V, and so no filter can converge to both x and y. Hence all filters have at most one limit.

Conversely, suppose that x and y do not have disjoint neighbourhoods. Then $N_x \cup N_y$ forms a subbase for a filter which converges to both x and y. So if every filter has at most one limit then X is hausdorff.

QED

So requiring X to be hausdorff is equivalent to requiring unique limits, which is a natural condition to impose (if we assume for the moment that filters are a good way of describing limits). In a hausdorff space we will write $\lim_F = x$ to mean x is the unique limit of F (note however that not all filters have limits).

Let X, Y be sets, F a filter on X and $g : X \to Y$. In general the set $\{g(U) : U \in F\}$ will not be a filter on Y. However because $g(U \cap V) \subseteq g(U) \cap g(V)$ it will be a subbase generating some filter. We shall denote the filter it generates by $g(F)$, in a mild abuse of notation that makes things much more convenient. A quick check shows that $V \in g(F)$ iff there exists $U \in F$ such that $g(U) \subseteq V$

Theorem 3 *Let X, Y be topological spaces with $x \in X$ and $g : X \to Y$. g is continuous at x iff whenever F is a filter such that $F \to x$ we have $g(F) \to g(x)$*

Proof:

Suppose g is continuous at x and let $F \to x$. Let V be a neighbourhood of $g(x)$. By continuity there is a neighbourhood U of x such that $g(U) \subseteq V$. But $U \in F$, so $g(U) \in g(F)$. Thus, as $g(F)$ is a filter, $V \in g(F)$. Hence $g(F) \to g(x)$.

Conversely, suppose whenever $F \to x$ we have $g(F) \to g(x)$. Then $g(N_x) \to g(x)$ by hypothesis. So for every V a neighbourhood of $g(x)$ we have $V \in g(F)$. Thus by our observation above there exists $U \in N_x$ such that $g(U) \subseteq V$. As N_x is the set of neighbourhoods of x, this is the statement that g is continuous at x.

QED

So filters capture the idea of continuity and limits quite well.

2 Filters and Product Spaces

If we have an indexed set of topological spaces $\{X_i : i \in I\}$ with filters F_i on X_i, we would like to put a filter F on the product space $\prod_{i \in I} X_i$ in such a way that convergence in the product behaves nicely.

We define it as follows, in order to make it consistent with the product topology:

Definition 3 *Consider the set of products $\prod_{i \in I} U_i$, where $U_i \in F_i$ and for all but finitely many i, $U_i = X_i$. This forms a subbasis, generating the product filter, which we will denote by F or $\prod_{i \in I} F_i$ (in what is again a slight abuse of notation).*

In the following theorems we will assume $X_i,\ F_i$ and F are as above.

Proposition 2 *Let π_i be the projection map onto the i coordinate. $\pi_i(F) = F_i$*

The proof is an easy exercise.

Theorem 4 *$F \to x$ in the product topology iff for each $i \in I$, $F_i \to X_i$.*

Proof:

One way is easy. $\pi_i(F) = F_i$, and π_i is continuous. Thus by Theorem 3, $F_i \to \pi_i(x) = x_i$.

Suppose for each i, $F_i \to x_i$. Let U be a neighbourhood of x. It is sufficient to consider $U = \prod_{i \in I} U_i$ with each U_i open, $U_i = X_i$ for all but finitely many i, and $x_i \in U_i$, as such sets must generate N_x (by the definition of the product topology). Then each $U_i \in F_i$ and for all but finitely many i, $U_i = X_i$, so by the definition of the product filter we have $U = \prod_{i \in I} U_i \in F$

QED

Corollary 1 *Let F be any filter on X, and let $F_i = \pi_i(F)$. $F \to x$ iff for each i, $F_i \to x_i$*

Proof: F contains the product filter $\prod_{i \in I} F_i$, so if $\prod_{i \in I} F_i \to x$ then $F \to x$.

QED

3 Filters and Compactness

With sequences we have the notion of sequential compactness - a topological space is said to be sequentially compact iff every sequence has a convergent subsequence. In metric spaces sequential compactness is equivalent to compactness. We would like to find a similar result for filters.

First note that filters work in the opposite direction to sequences (so to speak). Where in order to get convergence we would take a subsequence, for filters we want to extend to a larger filter (because the aim is to get it to include a neighbourhood filter of some point). So we say G extends F if $F \subseteq G$. It is trivial to check that if $F \to x$ and G extends F then $G \to X$.

The following theorem is then the analogy of sequential compactness.

Theorem 5 *Let X be a topological space. X is compact iff every filter can be extended to a convergent filter.*

Proof:

We shall show the equivalence with the closed set definition of compactness.

First suppose X is compact. Let F be a filter. Then $Z = \{\overline{U} \ : \ U \in F\}$ is a set of closed sets with the finite intersection property. Thus it has non-empty intersection. Let $x \in \bigcap Z$. Then, by hypothesis, for each $U \in F$ and each neighbourhood V of x, we have $U \cap V \neq \emptyset$, as $x \in (\overline{U})$. Thus $F \cup N_x$ forms a sub-base for a filter which extends F and converges to x.

Conversely, suppose every filter can be extended to a convergent filter. Let C be a collection of closed sets with the finite intersection property. Then C forms a subbase for a filter F. F can be extended to a filter G converging to a point x. Thus for each $U \in N_x$ and each $V \in C$ we must have $V \cap U \neq \emptyset$. Since V is closed, we thus must have $x \in V$. So $x \in \bigcap C$. Thus $\bigcap C \neq \emptyset$.

QED

Proposition 3 *Let X be a topological space. X is compact iff every ultrafilter converges.*

Proof:

This is an almost trivial consequence of the previous theorem. If X is compact then every ultrafilter can be extended to a convergent filter, but ultrafilters have no proper extension. Thus every ultrafilter converges. Conversely, every filter can be extended to an ultrafilter, so if every ultrafilter converges then every filter can be extended to a convergent filter.

QED

This now leads us to what is, I feel, a huge piece of evidence towards how useful filters are.

Theorem 6 *Tychnoff's Theorem*

Let $\{X_i \ : \ i \in I\}$ be an indexed collection of compact topological spaces. $X = \prod_{i \in I} X_i$ is compact in the product topology.

Proof:

We shall use the previous theorem. Let F be an ultrafilter on X. Then it is easy to check that $\pi_i(X)$ is an ultrafilter on X_i, and thus convergent by the compactness of X_i. For each i, pick a limit x_i of $\pi_i(X)$. Then by theorem 4, $F \to x$ as each $F_i \to x_i$. Thus every ultrafilter is convergent, and X is compact.

QED

It is also worth noting that in the case where each of the X_i is hausdorff, limits are unique so we don't need to pick the x_i - they are already picked for us. So Tychnoff's theorem for the special case of hausdorff spaces doesn't need the full power of the axiom of choice, and merely the much weaker ultrafilter theorem. (Whileas Tychonoff's theorem is equivalent to the axiom of choice).

4 Filters and Subspaces

When we have a topological space X and a set $A \subseteq X$ we want to be able to talk about filters in A converging to points $x \notin A$ in order to deal with limit points and the like. This is a fairly straightforward thing to do, and we define it in a reasonably obvious way.

Let F be a filter in A. Then F forms a filter base in X, and so generates a new filter F'. If $x \in X$ we say $F \to x$ iff $F' \to x$.

We must first show this is consistent:

Proposition 4 *Let A, X, F, F' be as above. Let $x \in A$. $F \to x$ in A iff $F' \to x$ in X.*

Proof:

Note that because F forms a filter base for F', we have $F' = \{U : \exists Y \in F \; Y \subseteq U\}$. Thus in particular $F \subseteq F'$

Suppose $F \to x$ in A. Let U be a neighbourhood of x in X. Then $U \cap A \in F$, as it is a neighbourhood of x in A, and thus $U \in F'$, as $U \cap A \subseteq U$. Hence $F' \to x$ in X.

Now suppose $F' \to x$ in X. Let U be a neighbourhood of x in X. Then $U \in F'$. So there exists $V \in F$ such that $V \subseteq U$. Hence $V \subseteq U \cap A$, as $V \subseteq A$. Thus $U \cap A \in F$, as F is a filter on A. Every neighbourhood of x in A is of the form $U \cap A$, so $F \to x$ in A.

QED

So, having shown that our extension of the definition is consistent, we now prove a vaguely interesting result:

Theorem 7 *$x \in X$ is a limit point of A iff there is a filter F on A such that $F \to x$*

Proof:

Suppose F is a filter on A such that $F \to x$. Then for every $U \in N_x$, we must have $U \cap A \in F$, and in particular $U \cap A \neq \emptyset$ (as $\emptyset \notin F$), and so x is a limit point of A.

Conversely, if x is a limit point of A then $\{U \cap A : U \in N_x\}$ is a filter on A converging to x.

QED

Trivial as all this is, it will come in useful when dealing with certain questions, often about metric spaces.

5 Filters and Metric Spaces

We will now use filters to provide a more elegant description of convergence and completeness in metric spaces.

Definition 4 *Let X be a metric space and F a filter on X. We say that F is cauchy if it contains arbitrarily small sets - i.e. $\forall \epsilon > 0 \;\; \exists M \in F \;\; diam(M) < \epsilon$.*

We say X is complete if every cauchy filter converges.

It is easy (but messy) to check that this is equivalent to the sequence definition of complete, but a proof shall not be included here.

Theorem 8 *Let F be a cauchy filter, and G a filter extending F with $G \to x$. Then $F \to x$.*

Proof:

For any $M \in F$ and every neighbourhood U of x we have $U, M \in G$ so $U \cap M \neq \emptyset$. Thus x is a limit point of M, so $d(x, M) \leq diam(M)$. Pick $\epsilon > 0$. There exists $M \in F$ with $diam(M) < \epsilon$. Then for all $y \in M$ we have $d(x, y) \leq diam(M) < \epsilon$. So $M \subseteq B(x, \epsilon)$. Hence $B(x, \epsilon) \in F$.

Thus $N_x \subseteq F$, so $F \to x$

QED

Corollary 2 *Let X be a compact metric space. X is complete.*

Theorem 9 *Let X be a metric space and $Y \subseteq X$. If Y is complete then it is closed. If X is complete and Y is closed then Y is complete.*

Proof:

If Y is not closed then let x be a limit point of Y such that $x \notin Y$. Then $N_{x,\ A}$ is a cauchy filter in Y which does not converge (by uniqueness of limits in $Y \cup \{x\}$)

Conversely, suppose Y is closed and X is complete. Let F be a cauchy filter on Y. This generates a cauchy filter G on X which projects onto A and has $G_A = F$. X if complete, so $G \to x$ for some x. But G projects onto A, so $G_A \to x$. Thus $F \to x$. Hence F converges. Thus Y is complete.

Proposition 5 *Let F be a cauchy filter and $g : X \to Y$ uniformly continuous. $g(F)$ is a cauchy filter.*

This is immediate from the definitions.

Theorem 10 *Let X, Y be metric spaces and Y complete. Let $W \subseteq X$ be dense and $g : W \to Y$ uniformly continuous. There exists a unique $f : X \to Y$*

Proof:

For $x \in W$ define $g(x) = f(x)$. For $x \notin W$, we know that $x \in \bar(W)$ so N_x projects onto W. Thus consider $N_{x,\ W}$. This is cauchy, so by uniform continuity we have that $g(N_{x,\ W})$ is cauchy, and thus convergent by the completeness of Y. This limit is unique, so define $g(x) = lim\ g(N_{x,\ W})$.

f is continuous on W, as it is equal to g there, and by construction if $x \notin W$ and $F \to x$ then $f(F_W) \to f(x)$. But $f(F_W) \subseteq f(F)$, so $f(F) \to f(x)$. Hence f is continuous at x.

QED