

Cardinal Arithmetic via the Upward Lowenheim Skolem Theorem

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Abstract

There are a number of theorems in cardinal arithmetic which essentially say ‘cardinal arithmetic is trivial’. These are straightforward but rather annoying to prove. In this paper I present a sequence of proofs of them that uses the fact that they are true for countably infinite sets together with the Upward Lowenheim Skolem Theorem in order to deduce that they in fact hold for sets of all cardinalities.

This is not expected or intended to be anything groundbreaking or new. It’s merely an amusing proof of some standard results.

Theorem 1 *The Upward Lowenheim Skolem Theorem*¹

Let L be a first order language with $|L| = \aleph_0$. Let T be a theory in L . If T has a countably infinite model then it has a model of every cardinality.

Various results we will be proving are known to be equivalent to the axiom of choice, and thus the upward lowenheim skolem theorem is also equivalent to the axiom of choice. Because I’m perverse, we will try to avoid using any other results that depend on AC (it’s essentially unnecessary anyway).

Theorem 2 *Let X be a set. We can write $X = X_1 \cup X_2$ where X_i are disjoint and have the same cardinality as X .*

Proof:

Let L be the first order language with unary functions f, g and a unary relation P . Let T be the theory consisting of the following sentences:

$$\begin{aligned}\forall x \quad f(f(x)) &= x \\ \forall x \quad \neg(P(x) \wedge P(f(x))) \\ \forall x \quad P(x) \vee P(f(x)) \\ \forall x, y \quad g(x) = g(y) &\implies x = y \\ \forall x \quad P(x) &\implies \exists y \quad g(y) = x\end{aligned}$$

¹The usual theorem stated is a stronger form than this, but this will suffice for our purposes.

This has a countable model, so has a model of every infinite cardinality.

Let M be a model of cardinality $|X|$. By taking an appropriate bijection we may assume the underlying set of M is X .

So we have functions $f, g : X \rightarrow X$ and a relation P on X such that if $I = \{x : P(x)\}$ we have the following properties:

1. f is a bijection.
2. $f|_I : I \rightarrow I^c$ is a bijection between I and I^c
3. g is a bijection from X to I .

So, in particular, I and I^c have the same cardinality (as f is a bijection between them) and I has the same cardinality as X (as g is a bijection between them). So upon letting $X_0 = I$ and $X_1 = I^c$ the result follows.

QED

Corollary 1 For an infinite set A , $|A| + |A| = |A|$.

Corollary 2 Let A, B be infinite with $|B| \leq |A|$. $|A| + |B| = |A|$

Proof:

We can split A up into $A_0 \cup A_1$ as above. We have an injection $h : B \rightarrow A$ and so an injection $h : B \rightarrow A_0$ (in a slight abuse of notation). Further we have an injection $g : A \rightarrow A_1$. So these give an injection from the disjoint union of A and B into A . So $|A| + |B| \leq |A|$. But clearly $|A| \leq |A| + |B|$, and so we have $|A| = |A| + |B|$ as required.

QED

We will prove a similar result for multiplication, though it will be more involved.

Theorem 3 Let X be an infinite set. There is a bijection $h : X^2 \rightarrow X$.

Proof:

Let L be the language with a binary function g . Let T be the following theory:

$$\begin{aligned} \forall x_i, y_i \quad g(x_1, y_1) = g(x_2, y_2) &\implies x_1 = x_2 \wedge y_1 = y_2 \\ \forall x \quad \exists u, v \quad g(u, v) = x & \end{aligned}$$

An interpretation of g on X is necessarily a bijection from X^2 to X .

Classically there is a bijection from N^2 to N (use your favourite example), so there is a countable model and thus a model of every cardinality. The result follows.

QED

Corollary 3 *Let A, B be infinite sets with $|B| \leq |A|$. $|A| \cdot |B| = |A|$*

Proof:

Let $f : B \rightarrow A$ and $h : A^2 \rightarrow A$ be injections. Then $(a, b) \rightarrow h(a, f(b))$ is an injection from $A \times B$ to A . There is clearly an injection from A to $A \times B$, so $|A| = |A \times B| = |A||B|$.