Surjective functions from $\{0,1\}^{\mathbb{N}}$

David MacIver

Abstract

It's a reasonably well known classic result in point set topology that every non-empty compact metric space admits a continuous surjection from the cantor set. When I realised I'd never actually seen a proof of this, I set out to prove it myself. It wasn't very hard, but the details were messy. Here's the result of my attempt to produce a nice, tidy proof of it.

Theorem 1 *Let X be a compact metric space. We can construct a canonical way of breaking up a closed set $F \subseteq X$ into n_F parts, $F_1, \dots, F_n$ such that each F_i is closed and has $diam(F_i) \leq diam(F)$, $\bigcup F_i = F$ and no strictly smaller collection of the F_i covers F.*

Fix a countable dense subset of X^1, say $\{a_n\}$. Let $\epsilon = \frac{1}{4} diam(F)$. Then $B(a_i, \epsilon) \cap F$ is an open cover of F, and so has a finite subcover. Let k be the first natural number such that $F \subseteq \cup_{i=1}^{k} B(a_i, \epsilon)$.

Now let $G_i = \overline{B}(a_i, \epsilon) \cap F$ for $i \leq k$.

G_i satisfies almost all the requirements for F_i, but it may be too big. This is not a problem. Suppose for some r, $G_r \subseteq \bigcup_{i \neq r} G_i$. Take the least such r and remove G_r from the sequence. Iterate this process, and because there are only finitely many terms it must eventually stop. This does not affect $\bigcup G_r$, or any of the other properties we want. Call the sequence of sets that this stops with $F_1, \dots F_n$

QED

Theorem 2 *We can explicitly construct a collection of functions $K_F : \{0,1\}^{n_F} \to P(F)$, where F ranges over the closed subsets of X, such that $\forall x\ K_F(x)$ is closed, non-empty and $diam(K_F(x)) \leq \frac{1}{2} diam(F)$.*

Proof:

Let $F_1, \dots F_n$ be as above. For $x \in \{0,1\}^{n_F}$ define r_x to be the smallest k such that $r_k = 0$, or if x is identically 1, $r_x = n$. Let $K_F(x) = F_{r_x}$. (Really this is a somewhat silly way of doing it. It's an artifact of the original way I had of doing this, but it's tidier than most of the other ways so I left it in).

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[1] This is the only place in this article we will use the axiom of choice

QED

Theorem 3 *Let X be a non-empty compact metric space. There is a continuous surjection $f : \{0,1\}^{\mathbb{N}} \to X$.*

Proof:

Consider pairs $(F, (x_n))$ where $F \subseteq X$ is closed and $(x_n) \in \{0,1\}^{\mathbb{N}}$. Let D be the set of such pairs.

Define a function $h : D \to D$ by:

Take (F, x). Let a be the finite sequence of the first n_F terms of x and $x = a \wedge y$.

Then let $h(F, x) = (K_F(a), y)$.

Now, given $x \in X$ consider the sequence of iterates $h^n(X, x) = (X_n, y_n)$.

By construction $X = X_0 \supset X_1 \supset X_2 \supset \ldots \neq \emptyset$, and $diam(X_n) \leq 2^{-n} diam(X)$. So $\bigcap X_n$ is a singleton, say t. Define $f(x) = t$.

It will also be convenient to have for each x the sequence $m_n = \sum_{i=0}^{n} n_{X_i}$.

Now that we have constructed f we show that it has the desired properties.

1. f is continuous.

   Let $x \in \{0,1\}^{\mathbb{N}}$ and $\epsilon > 0$. Let n be the least natural such that $2^{-n} < \epsilon$. Then the set of elements of $\{0,1\}^{\mathbb{N}}$ which agree with x on their first m_n terms is a neighbourhood of x. Let y be in this set. Then by construction of h, if $h^n(X, x) = (U, t)$ and $h^n(X, y) = (V, s)$ then $U = V$ (as at each stage of the iteration the sequences agree on the truncated terms). So in particular $f(x) \in U$ and $f(y) \in U$. So $d(f(y), f(x)) \leq 2^{-n} < \epsilon$. Hence f is continuous.

2. f is surjective.

   It is immediate by our construction and induction on n that if $h^n(X, t) = (U_n(t), t)$ then $\bigcup_{t \in \{0,1\}^{\mathbb{N}}} U_n(t) = X$.

   Thus let $x \in X$. There is some t such that $x \in U_n(t)$. But $h(t) \in U_n(t)$. Hence $d(x, h(t)) \leq 2^{-n}$.

   Now, $\{0,1\}^{\mathbb{N}}$ is compact, so the function $t \to d(x, f(t))$ attains its minimum value. We have just shown that this minimum value is $\leq 2^{-n}$ for every n, so it must be 0. Thus $\exists t \;\; d(x, f(t)) = 0$, i.e. $\exists t \;\; f(t) = x$.

   Hence f is surjective.

QED