Some basic analysis problems

These are of varying degrees of difficulty, but they all have elementary solutions. The guiding principle is that if you know enough to understand what the problem is asking then you should know enough to solve it. Where this is not the case I've made a note of it.

1 Real Analysis

1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) + f(2x) = 0$. Show f is constant.
   (b) Show by means of a counterexample the continuity hypothesis cannot be weakened to continuous except at one point.
2. Construct a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing and discontinuous at every irrational.
3. A function is 2 to 1 if for every y in it's image, $f^{-1}(y)$ consists of exactly two points (Note that it need not be surjective).Show there is no continuous 2 to 1 function $f : \mathbb{R} \to \mathbb{R}$.
4. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is irrational if and only if x is rational?
5. Show that there is a continuous surjective function $f : [0,1]\backslash\mathbb{Q} \to [0,1]\cap\mathbb{Q}$.
6. (a) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and satisfies $\forall x \in [0,1] \ \ |f(x)| \leq \int_0^x f(t)dt$. Show that f is identically 0.
   (b) What if f is merely riemann integrable?
7. Let $X \subseteq \mathbb{R}^n$ be such that for every $x, y \in X$ we have $||x - y|| \in \mathbb{Q}$. Show that X is countable.

   (Hint: Try showing it for $n = 2$ first and then generalising.)
8. (a) Let $f : [0,1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show that for all $n \in \mathbb{N}$ there exists $t \in [0,1]$ with $t + \frac{1}{n} \in [0,1]$ and $f(t) = f(t + \frac{1}{n})$.
   (b) Find all $x \in [0,1]$ such that for every such function there is a $t \in [0, m1]$ with $t + x \in [0,1]$ and $f(t + x) = f(t)$.
9. (a) Let X be a metric space such that every continuous $f : X \to \mathbb{R}$ is bounded. Show that X is compact.
   (b) Show that the same is not neccesarily true if we only assume X is a topological space. What if we also assume X is hausdorff?

   (The last bit is somewhat harder than the rest).

10. Let X be a compact metric space, $x_0 \in X$, $h : X \to X$ (not neccesarily continuous) and $g : X \to \mathbb{R}$ continuous. Suppose $h(x_0) = x_0$ and for any $x \neq x_0$ we have $d(h(x), x_0) < d(x, x_0)$ and $g(h(x)) \leq g(x)$. Show that for all $x \in X$ we have $g(x_0) \leq g(x)$.

11. Let X be a countable complete metric space. Show that the set of isolated points is dense.

    (This is slightly non-elementary.)

2 Complex Analysis

1. Let $f, g : \mathbb{C} \to \mathbb{C}$ be distinct analytic functions. Show that $\{z \ : \ f(z) = g(z)\}$ is countable.

2. (a) Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and let $f : D \to D$ be analytic. Show that if f is not the identity function then it has at most one fixed point.
   (b) Show by example that f need not have a fixed point. (It may help to consider a conformal transformation from D to some other region).
   (c) Show that for each $n \geq 0$ there is an analytic function $f : \mathbb{C} \to \mathbb{C}$ with n fixed points, and one with a countably infinite number of fixed points.
   (d) Is there $f : \mathbb{C} \to \mathbb{C}$ with uncountably many fixed points?

3. Let $f_n : \mathbb{C} \to \mathbb{C}$ be a sequence of distinct analytic functions. Show that there exists $z \in \mathbb{C}$ with $f_n(z) \neq f_m(z)$ whenever $n \neq m$.

3 Functional Analysis

1. Let X be any set and let F be a collection of functions from X into $\mathbb{R}$ which is closed under pointwise addition, multiplication and under multiplication by scalars. These operations make F into an algebra. Suppose $|\cdot|$ is a complete algebra norm on F. Show that $F \subseteq B(X)$ and there exists M such that for every $f \in F$ we have $||f||_\infty \leq |f| \leq M||f||_\infty$.

2. Let L^p_{loc} be the space of functions (up to equality a.e.) $f : \mathbb{R} \to \mathbb{R}$ to itself such that for every compact $K \subseteq \mathbb{R}$ we have $f|_K \in L^p(K)$. Define a topology on L^p_{loc} by saying that $f_n \to f$ if for every compact $K \subseteq \mathbb{R}$ we have $f_n|_K \to f|_K$ in the L^p norm. Show that this topology is not normable.

   (Hint: Consider the functions $f_n = I_{[n,\infty)}$)

3. Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Suppose $f_n : D \to \mathbb{C}$ are holomorphic and $f_n \to f$ in $L^2(X)$. Show that f is a.e. equal to a holomorphic function.

4. Let X, Y be Banach spaces and let $X_1, X_2 \subseteq X$ be closed subspaces with $X_1 + X_2 = X$. Let $T_i : X_i \to Y$ be continuous linear functionals such that $T_1|_{X_1 \cap C_2} = T_2|_{X_1 \cap C_2}$. Define $T : X \to Y$ by $T(x_1 + x_2) = T_1(x_1) + T_2(x_2)$ (where $x_i \in X_i$).

(a) Show that T is well defined.

(b) Find examples to show that for any $M > 0$ we can have $||T|| \geq M(||T_1|| + ||T_2||)$.

(c) Show that T is continuous.

(Hint: You may find the open mapping theorem useful).